MAU23101
Introduction to number theory
0 - Diophantine equations

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Module web page

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A Diophantine equation

Consider the equation

$$x^3 + y^3 + z^3 = 29.$$

Does it have solutions? How many?

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If we impose no restrictions on x, y, z, then the answer is easy: take any x and y, e.g. $x = 1$, $y = 2$, and then

$$z = \sqrt[3]{29 - x^3 - y^3} = \sqrt[3]{20}.$$

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Some solutions:

$$x = 1, \ y = 1, \ z = 3.$$

$$x = 4, \ y = -3, z = -2.$$

Consider now
$$x^3 + y^3 + z^3 = 30.$$

Does it have solutions?

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Does it have solutions?

$$x = 2220422932, \ y = -2218888517, \ z = -283059965.$$

This is the simplest solution!

That Diophantine equation again

Consider now
$$x^3 + y^3 + z^3 = 31.$$

Does it have solutions?

That Diophantine equation again

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Does it have solutions?
No, and that's very easy to prove!

That Diophantine equation again

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Does it have solutions?
No, and that's very easy to prove!

Same thing for
$$x^3 + y^3 + z^3 = 32.$$

Consider now
$$x^3 + y^3 + z^3 = 33.$$

Does it have solutions?

Consider now

$$x^3 + y^3 + z^3 = 33.$$

Does it have solutions?

No one knows!

Consider now
$$x^3 + y^3 + z^3 = 33.$$

Does it have solutions?

~~No one knows!~~

Until 2019, no one knew!